Fermi National Accelerator Laboratory

FERMILAB-PUB-89/248-T

December 13, 1989

Supersymmetry, Foldy-Wouthuysen transformation and stability of the Dirac sea.

Rodolfo Martínez[b], Matías Moreno [a,†] and Arturo Zentella[c]

a. Fermi National Accelerator Laboratory,

P. O. Box 500, Batavia, Ill. 60510, USA

b. Depto. de Física, Facultad de Ciencias,

Universidad Nacional Autónoma de México,

Ap. Postal 70-543, 04510 México, D. F., MEXICO

c. Instituto de Física, Universidad Nacional Autónoma de México,

Ap. Postal 20-364, 01000 México, D. F., MEXICO

Abstract. It is shown that for a large class of potential problems in the Dirac equation the positive and negative energy solutions do not mix even in the strong coupling limit. We prove that this property, which implies a stability of the Dirac sea, is connected to the presence of superalgebra operators in the Dirac equation. The exact and closed form for the Foldy-Wouthuysen hamiltonian which is used to prove this property is given. The potentials include the Dirac oscillator and the odd potentials and its non-abelian generalizations.

† On sabbatical leave from Instituto de Física, Universidad Nacional Autónoma de México.

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

In a recent work [1] it has been shown that a physical interpretation of the Dirac oscillator equation [2] can be readily obtained when a Foldy-Wouthuysen [3] transformation is performed for this equation. One remarkable outcome of this procedure is that, regardless of the intensity of the coupling, the Dirac oscillator potential does not mix the positive and negative energy states. Therefore, the definition of the vacuum state of this theory can be maintained when one switches the interaction on.

In this work we show that this property, which will be referred to as the stability of the Dirac sea, is shared by a large class of interactions. Futhermore, we show that after defining a generalization of supersymmetry for potential problems in the Dirac theory, this stability is the result of a custodial supersymmetry.

The Dirac oscillator is simply a Dirac equation in which the interaction with an external potential is introduced non-minimally with the substitution

$$\boldsymbol{p} \rightarrow \boldsymbol{p} - im\omega\beta\boldsymbol{r}, \tag{1}$$

where m is the particle mass and ω is the oscillator frequency; our conventions follow those of Ref. [4]. The resulting equation can be cast into a form that shows its covariance and in which the physical meaning of the potential is transparent [1] ,

$$(i\partial\!\!\!/ - m + \kappa\frac{e}{4m}\sigma^{\mu\nu}F_{\mu\nu})\Psi = 0, \tag{2}$$

where $\kappa = 2m^2\omega/e$ is an anomalous (chromo) magnetic moment. The external (chromo) electric, field from which the interaction in Eq. 1 might stem, can be given in terms of an explicit reference frame. Physically this is the frame in which the charge distribution is at rest. Defining this frame in terms of the velocity four vector u of its origin, the field from which the Dirac oscillator arises is given by

$$F^{\mu\nu} = u^\mu x^\nu - u^\nu x^\mu. \tag{3}$$

Let us consider a Dirac particle in a purely odd time independent external potential. Except for these requirements the interaction is quite general, in particular the potential can have internal (color-like) degrees of freedom. The Dirac equation hamiltonian for this system is:

$$H = \boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \gamma_5 \pi_5 + \beta m, \tag{4}$$

where

$$\pi_i = p_i + A_i(\boldsymbol{x}) + i\beta E_i(\boldsymbol{x}),\ i = 1, 2, 3, 5; \tag{5}$$

and $p_5 := 0$. These potentials can be classified by their Lorentz properties as follows:

1. E_5 is a pseudoscalar potential.
2. A_5 is a timelike part of an axial-vector potential.
3. $\boldsymbol{E}$ is the (chromo) electric field part of an anomalous magnetic moment interaction.
4. $\boldsymbol{A}$ is a minimal (chromo) magnetic interaction.

Let us study a Foldy-Wouthuysen transformation with generator

$$\begin{aligned} iS &= \beta \mathcal{O}_H \theta \\ &= \beta(\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \gamma_5 \pi_5)\theta \\ &= (\boldsymbol{\gamma} \cdot \boldsymbol{\pi} + \beta \gamma_5 \pi_5)\theta, \end{aligned} \tag{6}$$

where $\mathcal{O}_H$ is the odd part of the hamiltonian. The existence of a closed and simple canonical transformation, that decouples large from small components, depends on whether or not one can obtain an operator θ which commutes with β and $\mathcal{O}_H$. If such θ exists then the Foldy-Wouthuysen transformation is given by

$$e^{iS} = \cos(h\theta) + \boldsymbol{\alpha} \cdot \boldsymbol{\pi} h^{-1} \sin(h\theta). \tag{7}$$

with h an even root of $\mathcal{O}_H^2$.

Now, with the assumed θ properties it follows that

$$\{iS, H\} = 0; \tag{8}$$

and therefore the transformed Foldy-Wouthuysen hamiltonian is simply

$$H_{FW} = \boldsymbol{\alpha} \cdot \boldsymbol{\pi}(\cos(2h\theta) - \frac{m}{h}\sin(2h\theta)) + \beta(m\cos(2h\theta) + h\sin(2h\theta)), \tag{9}$$

with the selection

$$\tan(2h\theta) = \frac{h}{m}, \tag{10}$$

the transformed Foldy-Wouthuysen hamiltonian will be purely even. If one selects instead

$$\cot(2h\theta) = -\frac{h}{m}, \tag{11}$$

a Cini-Touschek [5] form will be gotten. Any of this two conditions for θ imply

$$\theta = \theta(h^2), \tag{12}$$

and are consistent with the required commutation conditions. The form of the resulting hamiltonian is

$$H_{FW} = \beta\sqrt{m^2 + h^2}, \tag{13}$$

which confirms the claim that the hamiltonian does not mix the positive and negative energy states. An explicit solution for h, a square root of h^2, is H_{FW} with the mass term set equal to zero. The hamiltonian in Eq. 13 can be obtained explicitly in terms of

$$h^2 = \boldsymbol{\pi}^\dagger \cdot \boldsymbol{\pi} + \pi_5^\dagger \pi_5 + [\pi_5^\dagger \boldsymbol{\pi} + \boldsymbol{\pi}^\dagger \pi_5 + i\boldsymbol{\pi}^\dagger \times \boldsymbol{\pi}] \cdot \boldsymbol{\sigma}, \tag{14}$$

the different terms in this formula can be expanded according to

$$\begin{aligned}
\boldsymbol{\pi}^\dagger \cdot \boldsymbol{\pi} &= (\mathbf{p}+\mathbf{A})^2 + \mathbf{E}^2 + \beta\nabla\cdot\mathbf{E} + i\beta\sum_{j=1}^{3}[A_j, E_j], && (15)\\
\pi_5^\dagger \pi_5 &= A_5^2 + E_5^2 + i\beta[A_5, E_5], && (16)\\
\pi_5^\dagger \boldsymbol{\pi} + \boldsymbol{\pi}^\dagger \pi_5 &= \{\mathbf{p}+\mathbf{A}, A_5\} + \{\mathbf{E}, E_5\} + && (17)\\
&\quad \beta((\nabla E_5) + i[\mathbf{A}, E_5] - i[A_5, \mathbf{E}]), \\
\boldsymbol{\pi}^\dagger \times \boldsymbol{\pi} &= -i(\nabla\times\mathbf{A}) + \mathbf{A}\times\mathbf{A} + \mathbf{E}\times\mathbf{E} + && (18)\\
&\quad i\beta(\mathbf{p}\times\mathbf{E} - \mathbf{E}\times\mathbf{p} + \mathbf{A}\times\mathbf{E} - \mathbf{E}\times\mathbf{A}).
\end{aligned}$$

Our result reduces in different limits to the special cases treated by Eriksen [6]. In particular, because we have not used the commutation properties of the potentials, all expressions can also be applied to non-abelian theories. The Dirac oscillator potential corresponds to the case $A_5 = E_5 = 0, \mathbf{A} = \mathbf{0}$ and $\mathbf{E} = \mathbf{r}$.

We remark two general features of this results. First, the Foldy-Wouthuysen hamiltonian reduces to the free particle hamiltonian in the zero coupling limit in a way that never mixes positive and negative energy states. Second, the energy gap between positive and negative energy states has a minimum value of $2m$ when the potentials are turned on. These are straightforward consequences of the form of the

Foldy-Wouthuysen hamiltonian in Eq. 13.

We will now show how this class of Dirac equations is closely related to a Schrödiger supersymmetric quantum mechanical problem. A Schrödinger equation is said to be supersymmetirc if the hamiltonian of the problem has the form [7,8,9]:

$$H_S = \{Q, Q^\dagger\}, \tag{19}$$

where the operators Q and $Q^\dagger$ are fermionic, *i.e.*,

$$Q^2 = 0 = Q^{\dagger 2}; \tag{20}$$

this implies that $[Q, H_S] = 0 = [Q^\dagger, H_S]$ and therefore $QQ^\dagger$ and $Q^\dagger Q$ are good quantum numbers . The energy eigenstates $|n>, |n_+>= Q^\dagger|n>$ and $|n_->= Q|n>$ are degenerate (but either $|n_+>$ or $|n_->$ vanish).

We now study a Dirac hamiltonian operator of the form

$$H = Q + Q^\dagger + \lambda, \tag{21}$$

with λ hermitian and such that

$$\{Q, \lambda\} = \{Q^\dagger, \lambda\} = 0; \tag{22}$$

this condition guarantees that $QQ^\dagger$ and $Q^\dagger Q$ commute with H and are also good quantum numbers for the Dirac equation in formula 21. Then we have

$$H^2 = \{Q, Q^\dagger\} + \lambda^2. \tag{23}$$

We can now construct a Foldy-Wouthuysen transformation for this supersymmetric problem; the unitary transformation is generated by

$$iS = \lambda(Q + Q^\dagger)\theta; \tag{24}$$

with a Foldy-Wouthuysen hamiltonian

$$H_{FW} = \frac{\lambda}{\sqrt{\lambda^2}}\sqrt{\{Q, Q^\dagger\} + \lambda^2}. \tag{25}$$

The Dirac oscillator [1,2] and Eriksen [6] are all special cases of this framework with

$$\lambda = \beta m, \tag{26}$$

$$Q = \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}, \tag{27}$$

$$Q^\dagger = \begin{pmatrix} 0 & M^\dagger \\ 0 & 0 \end{pmatrix}; \tag{28}$$

where the standard representation for the Dirac matrices was used, M is a quaternion of the form

$$M = \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{C}) + C_5; \tag{29}$$

and the C_i are arbitrary complex potentials. The relation of the C_i to the A_i and E_i is

$$C_i = A_i + iE_i. \tag{30}$$

In this work we analyzed the origin of the stability of the Dirac sea for a large class of potentials in Dirac equation, these potentials include the recently discovered Dirac oscillator [2] as well as the more general class of odd potentials first found by Eriksen [6]. The stability of the Dirac sea was proved using the existence of a Foldy-Wouthuysen transformation which we have explicitly constructed. The origin of this property was shown to be related to the way in which the Dirac equation is constructed out of superalgebra operators. This in turn lead to a generalization of the class of Dirac equations for which an exact and closed Foldy-Wouthuysen transformation can be obtained. Let us finally mention that further generalizations of the procedure presented in this work require a careful treatment. This is particularly certain for the time independence requirement and its connection with the unitarity of the Foldy-Wouthuysen transformation [10,11,12,13].

We would like to thank M. Torres for discussions on the subject of this work. One of us (M. M.) would also like to thank the hospitality he has received from Fermilab.

References

[1] M. Moreno and A. Zentella, *Jour. of Physics A: Math. Gen.* **22**, L821 (1989)

[2] M. Moshinsky and A. Szczepaniak, *Jour. of Physics A: Math. Gen.* **22**, L817 (1989)

[3] L. Foldy and S. Wouthuysen, *Phys. Rev.* **78**, 29 (1950).

[4] J. Bjorken and S. Drell, *Relativistic Quantum Mechanics,* Mc Graw-Hill, New York (1964).

[5] M. Cini and T. Touschek, *Nuovo Cimento* **7**, 422 (1958).

[6] E. Eriksen, *Phys. Rev.* **111**, 1011 (1958).

[7] C.A. Blockley and G. E. Stedman, *Eur. J. Phys.* **6**, 218 (1985).

[8] G. E. Stedman, *Eur. J. Phys.* **6,** 225 (1985).

[9] L.F. Urrutia and E. Hernández, *Phys. Rev. Lett.* **51,** 755 (1983).

[10] M.M. Nieto, *Phys. Rev. Lett.* **38,** 1042 (1977).

[11] T. Goldman, *Phys. Rev.* **D15,** 1063 (1977).

[12] J. Kupersztych, *Phys. Rev. Lett.* **42,** 483 (1979).

[13] L. Holster, *J. Math. Phys.* **30,** 1621 (1989).